

GAUSS MAP OF MINIMAL SURFACES WITH RAMIFICATION

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ABSTRACT. We prove that for any complete minimal surface M immersed in R^n , if in CP^{n-1} there are $q > n(n+1)/2$ hyperplanes H_j in general position such that the Gauss map of M is ramified over H_j with multiplicity at least e_j for each j and

$$\sum_{j=1}^q \left(1 - \frac{(n-1)}{e_j} \right) > n(n+1)/2,$$

then M must be flat.

1. INTRODUCTION

Let $x: M \rightarrow R^n$ be a (smooth, oriented) minimal surface immersed in R^n . Make M into a Riemann surface by decreeing that the 1-form $d\xi_1 + id\xi_2$ is of type $(1, 0)$, where (ξ_1, ξ_2) are any local isothermal coordinates of M . The Gauss map of x is defined to be

$$G: M \rightarrow Q_{n-2}(C) \subset CP^{n-1}, \quad G(z) = [(\partial x / \partial z)]$$

where $[(\cdot)]$ denotes the complex line in C^n through the origin and $(\cdot)$, $z = \xi_1 + i\xi_2$ is the holomorphic coordinate of M , and

$$Q_{n-2}(C) = \{(w_0 : \cdots : w_{n-1}; w_0^2 + \cdots + w_{n-1}^2 = 0\} \subset CP^{n-1}.$$

By the assumption of minimality of M , G is a holomorphic map of M into CP^{n-1} . It is a natural question to study the “value distribution” properties of the Gauss map G . Fujimoto (see [8]) has shown that the Gauss map of a nonflat minimal surfaces can omit at most $n(n+1)/2$ hyperplanes in general position in CP^{n-1} under the assumption that G is nondegenerate. The “nondegenerate” assumption was removed by the author (see [13]). The purpose of this paper is to study more general “value distribution” properties of the Gauss map. In particular, we study the Gauss map with ramification.

One says that G is *ramified over a hyperplane* $H = \{[w] \in CP^{n-1} : a_0 w_0 + \cdots + a_{n-1} w_{n-1} = 0\}$ with multiplicity at least e if all the zeros of the function $g_H = (G, A)$ have orders at least e , where $A = (a_0, \dots, a_{n-1})$. If the image of G omits H , we shall say that G is *ramified over H with multiplicity ∞* .

Our main result is the following:

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Theorem 1. Let M be a complete minimal surface immersed in R^n and assume that the Gauss map G of M is k -nondegenerate (that is $G(M)$ is contained in a k -dimensional linear subspace of CP^{n-1} , but none of lower dimension), $1 \leq k \leq n-1$. Let $H_i \subset CP^{n-1}$ be q hyperplanes in general position. If G is ramified over H_i with multiplicity at least e_i for each i . Then

$$\sum_{j=1}^q \left(1 - \frac{k}{e_j}\right) \leq (k+1) \left(n - \frac{k}{2} - 1\right) + n.$$

In particular, for any complete minimal surface M immersed in R^n , if in CP^{n-1} there are $q > n(n+1)/2$ hyperplanes in general position such that its Gauss map G is ramified over H_j with multiplicity at least e_j for each j and

$$\sum_{j=1}^q \left(1 - \frac{(n-1)}{e_j}\right) > n(n+1)/2,$$

then M must be flat.

In the case $m = 3$, $Q_1(C)$ can be identified with CP^1 . We have a better result.

Theorem 2. Let M be a complete minimal surface ($\subset R^3$). If there are $q(q > 4)$ distinct points $a_1, \dots, a_q \in CP^1$ such that the Gauss map of M is ramified over a_j with multiplicity at least e_j for each j and $\sum_{j=1}^q (1 - 1/e_j) > 4$, then M must be flat.

In particular, if the Gauss map omits five distinct points, then M must be flat.

2. FACTS ON HOLOMORPHIC CURVES INTO PROJECTIVE SPACES

We shall recall some known results in the theory of holomorphic curves.

(A) **Associated curve.** Let f be a nondegenerated holomorphic map of $\Delta_R: \{z: |z| < R\}$ into CP^k , where $0 < R \leq \infty$. Take a reduced representation $f = [Z_0 : \dots : Z_k]$, where $Z = (Z_0, \dots, Z_k): \Delta_R \rightarrow C^{k+1} - \{0\}$. Denote by $Z^{(j)}$ the j th derivative of Z and define

$$\Lambda_j = Z^{(0)} \wedge \dots \wedge Z^{(j)}: \Delta_R \rightarrow \bigwedge^{j+1} C^{k+1}$$

for $0 \leq j \leq k$. Evidently $\Lambda_{k+1} \equiv 0$.

let $P: \bigwedge^{j+1} C^{k+1} - \{0\} \rightarrow CP^{N_j}$ denote the canonical projection, where $N_j = \binom{k+1}{j+1} - 1$. The j th associated curve of f is the map $f_j = P(\Lambda_j)$.

It is well known [4] (also see [16]) that the pull-back Ω_j of the Fubini-study metric on CP^{N_j} by f_j is given by

$$(2.1) \quad \Omega_j = dd^c \log |\Lambda_j|^2 = \frac{i}{2\pi} \frac{|\Lambda_{j-1}|^2 |\Lambda_{j+1}|^2}{|\Lambda_j|^4} dz \wedge d\bar{z},$$

for $0 \leq j \leq k$ and by convention $\Lambda_{-1} \equiv 1$. Note that $\Omega_k \equiv 0$. It follows that

$$(2.2) \quad \text{Ric } \Omega_j = \Omega_{j-1} + \Omega_{j+1} - 2\Omega_j.$$

Take a hyperplane $H: (W, A) = 0$, where $A = (a_0, \dots, a_k)$ is a unit vector. Define

$$\varphi_j(H) = \frac{|\Lambda_j \vee A|^2}{|\Lambda_j|^2 |A|^2}.$$

Note that $0 \leq \varphi_j(H) \leq \varphi_{j+1}(H) \leq 1$ for $0 \leq j \leq k$ and $\varphi_k(H) = 1$.

We need the following well-known lemma (see [4, 16 and 17]).

Lemma 2.1. *Let H be a hyperplane in CP^k , then for any constant $N > 1$, for $0 \leq p \leq k-1$,*

$$(2.3) \quad dd^c \log \frac{1}{N - \log \varphi_p(H)} \geq \left\{ \frac{\varphi_{p+1}(H)}{\varphi_p(H)(N - \log \varphi_p(H))^2} - \frac{1}{N} \right\} \Omega_p,$$

on $\Delta_R - \{\varphi_p = 0\}$.

(B) Nochka weights and product to sum estimate. We consider q hyperplanes H_j ($1 \leq j \leq q$) in CP^k which are given by $H_j: (W, A_j) = 0$. According to Chen [2], we give the following definition.

Definition 2.2. We say that hyperplanes $H_1, \dots, H_q$ are in n -subgeneral position if, for every $1 \leq j_0 < \dots < j_n \leq q$, $A_{j_0}, A_{j_1}, \dots, A_{j_n}$ generate C^{k+1} .

In [11] (see also [2]), Nochka has given the following lemma to prove the Cartan conjecture.

Lemma 2.3. *Let $H_1, \dots, H_q$ be hyperplanes in CP^k located in the n -subgeneral position, where $q > 2n - k + 1$. Then there are some constants $\omega(1), \dots, \omega(q)$ and θ satisfying the following condition:*

- (i) $0 < \omega(j)\theta \leq 1$ ($1 \leq j \leq q$),
- (ii) $\theta(\sum_{j=1}^q \omega(j) - k - 1) = q - 2n + k - 1$,
- (iii) $1 \leq (n+1)/(k+1) \leq \theta \leq (2n-k+1)/(k+1)$,
- (iv) if $R \subset Q$ and $0 < \#R \leq n+1$, then $\sum_{j \in R} \omega(j) \leq d(R)$.

For the proof, see [2] or [11].

Definition 2.4. We call constants $\omega(j)$ ($1 \leq j \leq q$) and θ above Nochka weights and a Nochka constant for $H_1, \dots, H_q$ respectively.

Nachka weights are useful because of the following lemma.

Lemma 2.5. *Under the above assumptions. Let $E_1, \dots, E_q$ be a sequence of real numbers with $E_j \geq 1$ for all j . Then for any subset B of the set $\{1, 2, \dots, q\}$ with $0 < \#B \leq n+1$, there exists a subset C of B such that $\{A_j | j \in C\}$ is a base of the linear space spanned by $\{A_j | j \in B\}$ and*

$$\prod_{j \in B} E_j^{\omega(j)} \leq \prod_{j \in C} E_j,$$

where $\omega(j)$ are the Nochka weights associated to hyperplanes $H_j: (A_j, W) = 0$, $j = 1, 2, \dots, q$.

For the proof, see [2] or [11].

We also have the following product to sum estimate.

Lemma 2.6 (see Chen [2]). *Under the above assumptions. For $0 \leq p \leq k-1$, any constant $N > 1$, $1/q \leq \lambda_p \leq 1/(k-p)$, there exists a positive constant $c_p > 0$ only depends on p and the given hyperplanes such that*

$$(2.4) \quad \begin{aligned} c_p \prod_{j=1}^q \left(\frac{\varphi_{p+1}(H_j)^{\omega(j)}}{\varphi_p(H_j)} \frac{1}{(N - \log \varphi_p(H_j))^2} \right)^{\lambda_p} \\ \leq \sum_{j=1}^q \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N - \log \varphi_p(H_j))^2}, \end{aligned}$$

on $\Delta_R - \{\varphi_p = 0\}$.

3. METRICS WITH NEGATIVE CURVATURE

We retain the notation of the last section. Let $f: \Delta_R \rightarrow \mathbb{C}P^k$ be a nondegenerate holomorphic map. Take a reduced representation $f = [Z_0 : \cdots : Z_k]$ where $Z = (Z_0, \dots, Z_k): \Delta_R \rightarrow \mathbb{C}^{k+1} - \{0\}$ is a holomorphic map. Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{C}P^k$ located in n -subgeneral position. Let $\omega(j)$ be their Nochka weights.

Let f be ramified over H_j with multiplicity at least e_j for each j . Assume that

$$\sum_{j=1}^q \left(1 - \frac{k}{e_j} \right) > 2n - k + 1,$$

we shall construct a continuous pseudo-metric on Δ_R such that its Gauss curvature is less than or equal to -1 . So that we can use Schwarz lemma to obtain our main inequality.

Let $\Omega_p = \frac{i}{2\pi} h_p(z) dz \wedge d\bar{z}$. Let

$$(3.1) \quad \sigma_p = c_p \prod_{j=1}^q \left[\left(\frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)} \right)^{\omega(j)(1-k/e_j)} \frac{1}{(N - \log \varphi_p(H_j))^2} \right]^{\lambda_p} h_p.$$

Where c_p is the constant in the product to sum estimate,

$$\lambda_p = 1 / \left((k-p) + (k-p)^2 \frac{2q}{N} \right),$$

and $N > 1$.

We take the geometric mean of the σ_p and define

$$(3.2) \quad \Gamma = \frac{i}{2\pi} c \prod_{p=0}^{k-1} \sigma_p^{\beta_k/\lambda_p} dz \wedge d\bar{z}.$$

where $\beta_k = 1/(\sum_{p=0}^{k-1} \lambda_p^{-1})$, and $c = 2(\prod_{p=0}^{k-1} \lambda_p^{\lambda_p^{-1}})^{\beta_k}$.

Let

$$(3.3) \quad \Gamma = \frac{i}{2\pi} h(z) dz \wedge d\bar{z}.$$

We now compute $h(z)$. By (3.1) and (3.2), we have

$$(3.4) \quad h(z) = c \left[\prod_{j=1}^q \frac{k}{\varphi_0(H_j)^{\omega(j)(1-1/e_j)\beta_k}} \prod_{p=0}^{k-1} \frac{h_p^{\beta_k/\lambda_p}}{(N - \log \varphi_p(H_j))} \right].$$

By (2.1),

$$h_p^{1/\lambda_p} = \left(\frac{|\Lambda_{p-1}|^2 |\Lambda_{p+1}|^2}{|\Lambda_p|^4} \right)^{(k-p)+(k-p)^2 2q/N},$$

so

$$\prod_{p=0}^{k-1} h_p^{1/\lambda_p} = |\Lambda_0|^{-2(k+1)-(k^2+2k-1)4q/N} |\Lambda_1|^{8q/N} \cdots |\Lambda_{k-1}|^{8q/N} |\Lambda_k|^{2+4q/N}.$$

Notice that $|\Lambda_0| = |Z|$, and $\varphi_0(H_j) = |(Z, A_j)|^2/|Z|^2$, therefore

$$(3.5) \quad h(z) = c \left[\frac{|Z|^{\sum_{j=1}^q \omega(j)(1-k/e_j) - (k+1) - (k^2+2k-1)2q/N} (|\Lambda_1| \cdots |\Lambda_{k-1}|)^{4q/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1-k/e_j)} \prod_{p=0}^{k-1} (N - \log \varphi_p(H_j))} \right]^{2\beta_k}.$$

Lemma 3.1. *The function*

$$\frac{|\Lambda_k|}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1-k/e_j)}}$$

is continuous on Δ_R .

Proof. We shall prove that the function

$$P = \left[\frac{|\Lambda_k|^2}{\prod_{j=1}^q \varphi_0(H_j)^{\omega(j)(1-k/e_j)}} \right]^e$$

is continuous where $e = e_1 \cdots e_q$. Lemma 3.1 follows from this. According to the expression of $P(z)$, we only need to consider the points at which (Z, A_j) vanishes. For zero point z_0 of (Z, A_j) , since f is ramified over H_j with multiplicity at least e_j for each j , we have

$$(Z, A_j) = (z - z_0)^{\nu_j} Q_j(z)$$

where $Q_j(z_0) \neq 0$, and $\nu_j \geq e_j$ or $\nu_j = 0$. The n -subgeneral position implies that, at each point z , there are at most n of hyperplanes H_j , such that $(Z(z), A_j) = 0$. Thus there exists a constant c_0 (depending only on the given hyperplanes) such that

$$\#B = \#\{j \mid |(Z(z), A_j)|/|A_j||Z(z)| \leq c_0\} \leq n.$$

Let $E_j = 1/\varphi_0(H_j)^{\omega(j)(1-k/e_j)}$, then $E_j \leq 1$. If $j \notin B$, then $\varphi_0(H_j) > c_0$, so $E_j \leq c_1$ (depending only on the given hyperplanes).

Applying Lemma 2.5 with E_j above, we obtain

$$\begin{aligned} \frac{|\Lambda_k|^2}{\prod_{j=1}^q \varphi_0(H_j)^{\omega(j)(1-k/e_j)}} &\leq c_2 \frac{|\Lambda_k|^2}{\prod_{j \in B} \varphi_0(H_j)^{\omega(j)(1-k/e_j)}} \\ &\leq c_2 \frac{|\Lambda_k|^2}{\prod_{j \in C} \varphi_0(H_j)^{(1-k/e_j)}}. \end{aligned}$$

We may assume the index set $C = \{1, 2, \dots, l\}$ and $l \leq k+1$, therefore

$$\left[\prod_{j \in C} (Z(z), A_j)^{(1-k/e_j)} \right]^e = (z - z_0)^b R(z)$$

where $b = \sum_{j=1}^l e\nu_j(1 - k/e_j)$ and R is a holomorphic function such that $R(z_0) \neq 0$. Since

$$\begin{aligned} |\Lambda_k| &= \det \begin{vmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_k \\ Z'_0 & Z'_1 & Z'_2 & \cdots & Z'_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_0^{(k)} & Z_1^{(k)} & Z_2^{(k)} & \cdots & Z_k^{(k)} \end{vmatrix} \\ &= \det \begin{vmatrix} (Z, A_1) & (Z, A_2) & (Z, A_3) & \cdots \\ (Z, A_1)' & (Z, A_2)' & (Z, A_3)' & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (Z, A_1)^{(k)} & (Z, A_2)^{(k)} & (Z, A_3)^{(k)} & \cdots \end{vmatrix}, \end{aligned}$$

we have $\Lambda_k = (z - z_0)^\nu S(z)$, where $\nu = \nu_1 + \nu_2 - 1 + \cdots + \nu_l - k$ and S is a holomorphic function. Hence we obtain

$$P(z) \leq |(z - z_0)^{2p} T(z)|,$$

where

$$p = \frac{ek}{e_1} + \frac{e}{e_2}(k\nu_2 - e_2) + \frac{e}{e_3}(k\nu_3 - 2e_3) + \cdots + \frac{e}{e_l}(k\nu_l - (l-1)e_l) \geq 0,$$

and $T(z)$ is continuous at z_0 . Therefore $P(z)$ is bounded around z_0 . Therefore $P(z)$ is continuous. Q.E.D.

Lemma 3.2. If $\sum_{j=1}^q (1 - k/e_j) \geq 2n - k + 2$, and

$$2q/N < \left(\sum_{j=1}^q \omega(j)(1 - k/e_j) - (k+1) \right) / (k^2 + 2k),$$

we have

(i) $\text{Ric} \Gamma \geq \Gamma$ on $\Delta_R - \bigcup \{\varphi_0(H_j) = 0\}$.

(ii) Γ is a continuous pseudo-metric on Δ_R .

Proof. From (3.3) and (3.4) it follows that

$$\begin{aligned} \text{Ric} \Gamma &= -\beta_k \sum_{j=1}^q \omega(j) \left(1 - \frac{k}{e_j} \right) dd^c \log \varphi_0(H_j) \\ &\quad + \beta_k \sum_{j=1}^q \sum_{p=0}^{k-1} dd^c \log(1/(N - \log \varphi_p(H_j)))^2 \\ &\quad + \beta_k \sum_{p=0}^{k-1} (1/\lambda_p) \text{Ric} \Omega_p. \end{aligned}$$

By Lemma 2.1, (2.2), and that $dd^c \log \varphi_0(H_j) = -\Omega_0$, we have

$$\begin{aligned} \text{Ric} \Gamma &\geq \beta_k \left(\sum_{j=1}^q \omega(j) \left(1 - \frac{k}{e_j} \right) \Omega_0 + 2 \sum_{j=1}^q \sum_{p=0}^{k-1} \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N - \log \varphi_p(H_j))^2} \Omega_p \right. \\ &\quad \left. - \frac{2q}{N} \sum_{p=0}^{k-1} \Omega_p + \sum_{p=0}^{k-1} \left[(k-p) + (k-p)^2 \frac{2q}{N} \right] \{ \Omega_{p+1} - 2\Omega_p + \Omega_{p-1} \} \right). \end{aligned}$$

Using Lemma 2.6, we obtain

$$\begin{aligned} & \sum_{j=1}^q \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N - \log \varphi_p(H_j))^2} \Omega_p \\ & \leq c_p \left[\prod_{j=1}^q \left(\frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)} \right)^{\omega(j)} \frac{1}{(N - \log \varphi_p(H_j))^2} \right]^{\lambda_p} \Omega_p \\ & \geq \frac{i}{2\pi} \sigma_p dz \wedge d\bar{z}. \end{aligned}$$

We also notice that $\Omega_k = 0$ so that

$$\sum_{p=0}^{k-1} (k-p)(\Omega_{p+1} - 2\Omega_p + \Omega_{p-1}) = -(k+1)\Omega_0$$

and therefore

$$\begin{aligned} \text{Ric} \Gamma \geq \beta_k & \left(\sum_{j=1}^q \omega(j) \left(1 - \frac{k}{e_j} \right) \Omega_0 + 2 \frac{i}{2\pi} \sum_{p=0}^{k-1} \sigma_p dz \wedge d\bar{z} - (k+1)\Omega_0 - (k^2 + 2k) \frac{2q}{N} \Omega_0 \right. \\ & \left. + \sum_{p=1}^{k-2} [(k-p+1)^2 - 2(k-p)^2 + (k-p-1)^2 - 1] \frac{2q}{N} \Omega_p + \frac{2q}{N} \Omega_{k-1} \right). \end{aligned}$$

The following is an elementary inequality:

For all the positive numbers $x_1, \dots, x_n$ and $a_1, \dots, a_n$,

$$(3.6) \quad a_1 x_1 + \dots + a_n x_n \geq (a_1 + \dots + a_n) (x_1^{a_1} \dots x_n^{a_n})^{1/(a_1 + \dots + a_n)}.$$

Letting $a_p = \lambda_p^{-1}$ in (3.6), we have

$$\sum_{p=0}^{k-1} \sigma_p \geq \frac{c}{2\beta_k} \sum_{p=0}^{k-1} \sigma_p^{\beta_k/\lambda_p}$$

and therefore

$$\begin{aligned} \text{Ric} \Gamma \geq \beta_k & \left(\left(\sum_{j=1}^q \omega(j) \left(1 - \frac{k}{e_j} \right) - (k+1) - (k^2 + 2k) \frac{2q}{N} \right) \Omega_0 \right. \\ & \left. + \sum_{p=0}^{k-2} \frac{2q}{N} \Omega_p + \frac{2q}{N} \Omega_{k-1} \right) + \Gamma. \end{aligned}$$

By Lemma 2.2, we find

$$\begin{aligned} \theta \left(\sum_{j=1}^q \omega(j) \left(1 - \frac{k}{e_j} \right) - k - 1 \right) &= \theta \left(\sum_{j=1}^q \omega(j) - k - 1 \right) - \frac{\sum_{j=1}^q \omega(j) \theta k}{e_j} \\ &= q - 2n + k - 1 - \frac{\sum_{j=1}^q \omega(j) \theta k}{e_j} \geq q - 2n + k - 1 - \frac{k}{e_j} \\ &= \sum_{j=1}^q \left(1 - \frac{k}{e_j} \right) - 2n + k - 1 > 0 \end{aligned}$$

and $\theta > 0$, so

$$\sum_{j=1}^q \omega(j) \left(1 - \frac{k}{e_j}\right) - (k+1) > 0.$$

This implies $\text{Ric} \Gamma \geq \Gamma$. Thus (i) is satisfied.

(ii) follows from Lemma 3.1, (3.3) and (3.5). Q.E.D.

We recall the following generalization of the Schwarz lemma.

Lemma 3.3. *Let $\Gamma = \frac{i}{2\pi} h(z) dz \wedge d\bar{z}$ be a continuous pseudo-metric on Δ_R whose curvature is bounded above by a negative constant. Then, for some positive c_0 , $h(z) \leq c_0(2R/(R^2 - |z|^2))^2$. For the proof, see [1, pp. 12–14].*

The purpose of this section is to obtain the following lemma.

Main Lemma. *Let $f = [Z_0 : \cdots : Z_k] : \Delta_R \rightarrow CP^k$ be a nondegenerate holomorphic map, $H_1, \dots, H_q$ be hyperplanes in CP^k in n -subgeneral position, $\omega(j)$ be their Nochka weights. Let $H_j : (W, A_j) = 0$ and $Z = (Z_0, \dots, Z_k)$. If f is ramified over H_j with multiplicity at least e_j for each j , $\sum_{j=1}^q (1 - k/e_j) > 2n - k + 1$ and $N > 2q(k^2 + 2k)/(\sum_{j=1}^q \omega(j)(1 - k/e_j) - (k+1))$, then there exists a positive constant c such that*

$$\begin{aligned} |Z| & \sum_{j=1}^q \omega(j)(1 - k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N \frac{\prod_{p=0}^{k-1} \prod_{j=1}^q |\Lambda_p \vee A_j|^{4/N} |\Lambda_k|^{1+2q+N}}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1 - k/e_j)}} \\ & \leq c \left(\frac{2R}{(R^2 - |z|^2)} \right)^{k(k+1)2 + \sum_{p=0}^{k-1} (k-p)^2 2q/N} \end{aligned}$$

Proof. Using the above Schwarz lemma for Γ , we obtain

$$h(z) \leq c_0(2R/(R^2 - |z|^2))^2.$$

So by (3.5) we have

$$\begin{aligned} (3.7) \quad |Z| & \sum_{j=1}^q \omega(j)(1 - k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N \frac{(|\Lambda_1| \cdots |\Lambda_{k-1}|)^{4q/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1 - k/e_j)} \prod_{p=0}^{k-1} (N - \log \varphi_p(H_j))} \\ & \leq c_0 \left(\frac{2R}{R^2 - |z|^2} \right)^{1/\beta_k}. \end{aligned}$$

Set $K := \sup_{0 < x \leq 1} x^{2/N} (N - \log x)$. Since $\varphi_p(H_j) < 1$ for all p and j we have

$$\frac{1}{(N - \log \varphi_p(H_j))} \geq \frac{1}{K} \varphi_p(H_j)^{2/N} = \frac{1}{K} \frac{|\Lambda_p \vee A_j|^{4/N}}{|\Lambda_p|^{4/N}}.$$

Substituting these into (3.7), we obtain the desired conclusion.

4. PROOF OF THEOREM 1

The proof of Theorem 1 basically follows the argument in [13] using the main lemma (see also the arguments in [6, 7 and 8]). We include our proof here for the convenience of the reader.

We may assume M is simply connected, otherwise we consider its universal covering. By Koebe's uniformization theorem, M is biholomorphic to C or to the unit disc. For the case $M = C$, Nochka (see [10], also see [16]) proved

that if a k -nondegenerate holomorphic map from C to CP^{n-1} is ramified over hyperplanes H_j ($1 \leq j \leq q$) with multiplicity at least e_j , where H_j are in general position, then

$$\sum_{j=1}^q \left(1 - \frac{k}{e_j}\right) \leq 2(n-1) - k + 1;$$

in this case our Theorem 1 is true. For our purpose it suffices to consider the case $M = \Delta$.

We first prove the first part of Theorem 1.

Assume the first part of Theorem 1 is not true, namely G is ramified over hyperplanes $H_1, \dots, H_q$ in CP^{n-1} in general position with multiplicity e_j and

$$(4.1) \quad \sum_{j=1}^q (1 - k/e_j) > (k+1)(n - k/2 - 1) + n.$$

Let $\omega(j)$ be Nochka weights of $\{H_j\}$. Because G is k -nondegenerate, we may assume $G(\Delta) \subset CP^k$, so that $G = [g_0 : \dots : g_k] : \Delta \rightarrow CP^k$ is nondegenerate. We consider hyperplanes $H_j \cap CP^k$, obviously these hyperplanes are in $(n-1)$ -subgeneral position in CP^k . For the convenience, we still denote these hyperplanes by $\{H_j\}$.

Let $\tilde{G} = (g_0, \dots, g_k) : \Delta \rightarrow CP^{k+1} - \{0\}$; then the metric ds^2 on M induced from the standard metric on R^n is given by

$$(4.2) \quad ds^2 = 2|\tilde{G}|^2 |dz|^2.$$

By Lemma 2.2,

$$q - 2(n-1) + k - 1 = \theta \left(\sum_{j=1}^q \omega(j) - k - 1 \right), \quad 0 < \omega(j)\theta \leq 1,$$

and

$$\theta \leq \frac{2(n-1) - k + 1}{k+1} = \frac{2n - k - 1}{k+1},$$

so

$$\begin{aligned} 2 \left(\sum_{j=1}^q \omega(j) \left(1 - \frac{k}{e_j}\right) - k - 1 \right) &= \frac{2\theta \left(\sum_{j=1}^q \omega(j) - k - 1 \right)}{\theta} - 2 \sum_{j=1}^q \frac{k\omega(j)\theta}{\theta e_j} \\ &= \frac{2(q - 2n + k + 1)}{\theta} - 2 \sum_{j=1}^q \frac{k\omega(j)\theta}{\theta e_j} \\ &\geq \frac{2(q - 2n + k + 1)}{\theta} - 2 \sum_{j=1}^q \frac{k}{\theta e_j} \\ &= \frac{2 \left(\sum_{j=1}^q (1 - k/e_j) - 2n + k + 1 \right)}{\theta} \\ &\geq \frac{2 \left(\sum_{j=1}^q (1 - k/e_j) - 2n + k + 1 \right) (k+1)}{(2n - k - 1)} \\ &> k(k+1) \quad (\text{by (4.1)}). \end{aligned}$$

Consider numbers

$$(4.3) \quad \rho = \frac{k(k+1)/2 + \sum_{p=0}^{k-1} (k-p)^2 2q/N}{\sum_{j=1}^q \omega(j)(1-k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N},$$

$$(4.4) \quad \gamma = \frac{k(k+1)/2 + qk(k+1)/N + 2q/N \sum_{p=0}^{k-1} p(p+1)}{\sum_{j=1}^q \omega(j)(1-k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N},$$

$$(4.5) \quad \delta = \frac{1}{(1-\gamma) \left(\sum_{j=1}^q \omega(j)(1-k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N \right)}.$$

Choose some N with

$$\begin{aligned} & \frac{\sum_{j=1}^q \omega(j)(1-k/e_j) - (k+1) - k(k+1)/2}{k^2 + 2k - 1 + \sum_{p=0}^k (k-p)^2} \\ & > 2q/N > \frac{\sum_{j=1}^q \omega(j)(1-k/e_j) - (k+1) - k(k+1)/2}{1/q + (k^2 + 2k - 1) + k(k+1)/2 + \sum_{p=0}^{k-1} p(p+1)} \end{aligned}$$

so that

$$(4.6) \quad 0 < \rho < 1, \quad 2\delta/N > 1.$$

Consider the open subset

$$M' = M - \left(\{ \tilde{G}_k = 0 \} \bigcup_{1 \leq j \leq q, \ 0 \leq p \leq k-1} \{ \tilde{G}_p \vee A_j = 0 \} \right)$$

of M and define the function

$$v = \left(\frac{\prod_{j=1}^q |(G, A_j)|^{\omega(j)(1-k/e_j)}}{\prod_{p=0}^{k-1} \prod_{j=1}^q |\tilde{G}_p \vee A_j|^{4/N} |\tilde{G}_k|^{1+2q/N}} \right)^\delta$$

on M' , where $\tilde{G}_p = \tilde{G}^{(0)} \wedge \dots \wedge \tilde{G}^{(p)}$. By Lemma 3.1, $\nu(z)$ is strictly positive and continuous on M' .

Let $\pi: \tilde{M}' \rightarrow M'$ be the universal covering of M' . Since $\log v \circ \pi$ is harmonic on $\tilde{M}'$ by the assumption, we can take a holomorphic function β on $\tilde{M}'$ such that $|\beta| = v \circ \pi$. Without loss of generality, we may assume that M' contains the origin 0 of C . As in Fujimoto's paper [6, 7, 8], for each point $\tilde{p}$ of $\tilde{M}'$ we take a continuous curve $\gamma_{\tilde{p}}: [0, 1] \rightarrow M'$ with $\gamma_{\tilde{p}}(0) = 0$ and $\gamma_{\tilde{p}}(1) = \pi(\tilde{p})$, which corresponds to the homotopy class of $\tilde{p}$. Let $\tilde{0}$ denote the point corresponding to the constant curve 0. Set

$$w = F(\tilde{p}) = \int_{\gamma_{\tilde{p}}} \beta(z) dz.$$

Then F is a single-valued holomorphic function on M' satisfying the condition $F(\tilde{0}) = 0$ and $dF(\tilde{p}) \neq 0$ for every $\tilde{p} \in \tilde{M}'$. Choose the largest R ($\leq \infty$) such that F maps an open neighborhood U of $\tilde{0}$ biholomorphically onto an open disc Δ_R in C , and consider the map $B = \pi \circ (F|U)^{-1}: \Delta_R \rightarrow M'$. By the Liouville theorem, $R = \infty$ is impossible.

By the definition of $w = F(z)$ we have

$$(4.7) \quad |dw/dz| = v(z).$$

For each point $a \in \partial\Delta$ consider the curve

$$L_a: w = ta, \quad 0 \leq t < 1,$$

and the image Γ_a of L_a by B . We shall show that there exists a point a_0 in $\partial\Delta_R$ such that Γ_{a_0} tends to the boundary of M . To this end, we assume the contrary. Then, for each $a \in \partial\Delta_R$, there is a sequence $\{t_\nu: \nu = 1, 2, \dots\}$ such that $\lim_{\nu \rightarrow \infty} t_\nu = 1$ and $z_0 = \lim_{\nu \rightarrow \infty} B(t_\nu a)$ exist in M . Suppose that $z_0 \notin M'$. Let $\delta_0 = 4\delta/N > 1$. Then by Lemma 3.1, we have

$$\liminf_{z \rightarrow z_0} |\tilde{G}_k|^{\delta_0} \prod_{1 \leq j \leq q, 1 \leq p \leq k-1} |\tilde{G}_p \vee A_j|^{2\delta_0} \cdot v > 0.$$

If $\tilde{G}_k(z_0) = 0$ or $|\tilde{G}_p \vee A_j|(z_0) = 0$ for some p and j , we can find a positive constant c such that $v \geq c/|z - z_0|^{\delta_0}$ in a neighborhood of z_0 , so that we obtain

$$\begin{aligned} R &= \int_{L_a} |dw| = \int_{L_a} \left| \frac{dw}{dz} \right| |dz| = \int v(z) |dz| \\ &\geq c \int_{\Gamma} \frac{1}{|z - z_0|^{\delta_0}} |dz| = \infty. \end{aligned}$$

This is a contradiction. Therefore, we have $z_0 \in M'$.

Take a simply connected neighborhood V of z_0 which is relatively compact in M' . Set $C' = \min_{z \in V} v(z) > 0$. Then $B(ta) \in V$ ($t_0 < t < 1$) for some t_0 . In fact, if not, Γ_a goes and returns infinitely often from ∂V to a sufficiently small neighborhood of z_0 and so we get the absurd conclusion

$$R = \int_{L_a} |dw| \geq c' \int_{\Gamma_a} |dz| = \infty.$$

By the same argument, we can easily see that $\lim_{t \rightarrow 1} B(ta) = z_0$. Since π maps each connected component of $\pi^{-1}(V)$ biholomorphically onto V , there exists the limit

$$\tilde{p}_0 = \lim_{t \rightarrow 1} (F|U)^{-1}(ta) \in \tilde{M}'.$$

Thus $(F|U)^{-1}$ has a biholomorphic extension to a neighborhood of a . Since a is arbitrarily chosen, F maps an open neighborhood of $\bar{U}$ biholomorphically onto an open neighborhood of $\bar{\Delta}_R$. This contradicts the property of R . In conclusion, there exists a point $a_0 \in \partial\Delta_R$ such that Γ_{a_0} tends to the boundary of M .

Our goal is to show that Γ_{a_0} has finite length, contradicting the completeness of the given minimal surface M .

By (4.7) we obtain $|dw/dz| = v(z)$. So

$$\begin{aligned} (4.8) \quad \left| \frac{dw}{dz} \right| &= |v(z)|^{1-\gamma} \left| \frac{dw}{dz} \right|^\gamma \\ &= \left(\frac{\prod_{j=1}^q |(\tilde{G}, A_j)|^{\omega(j)(1-k/e_j)}}{\prod_{p=0}^{k-1} \prod_{j=1}^q |\tilde{G}_p \vee A_j|^{4/N} |\tilde{G}_k|^{1+2q/N}} \right)^{1/(\sum \omega(j)(1-k/e_j) - (k+1) - (k^2+2k-1)2q/N)} \left| \frac{dw}{dz} \right|^\gamma. \end{aligned}$$

Let $Z(w) = \tilde{G} \circ B(w)$, $Z_0(w) = g_0 \circ B(w)$, $\dots$, $Z_k(w) = g_k \circ B(w)$. Then because

$$Z \wedge Z' \wedge \dots \wedge Z^{(p)} = (\tilde{G} \wedge \dots \wedge \tilde{G}^{(p)}) \left(\frac{dz}{dw} \right)^{p(p+1)/2},$$

it is easy to see that

$$(4.9) \quad \left| \frac{dw}{dz} \right| = \left(\frac{\prod_{j=1}^p |(Z, A_j)|^{\omega(j)(1-k/e_j)}}{\prod_{p=0}^{k-1} \prod_{j=1}^q |\Lambda_p \vee A_j|^{4/N} |\Lambda_k|^{1+2q/N}} \right)^{1/(\sum \omega(j)(1-k/e_j) - (k+1) - (k^2+2k-1)2q/N)},$$

where $\Lambda_p = Z^{(0)} \wedge \dots \wedge Z^{(p)}$.

On the other hand, the metric on Δ_R induced from $ds^2 = 2|\tilde{G}|^2|dz|^2$ through B is given by

$$(4.10) \quad B^*ds^2 = 2|\tilde{G}(B(w))|^2 \left| \frac{dz}{dw} \right|^2 |dw|^2.$$

Combining (4.7) and (4.8) gives

$$B^*ds = 2|Z| \left(\frac{\prod_{p=0}^{k-1} \prod_{j=1}^q |\Lambda_p \vee A_j|^{4/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1-k/e_j)}} \right)^{1/(\sum \omega(j)(1-k/e_j) - (k+1) - (k^2+2k-1)2q/N)} |dw|.$$

Using the main lemma, we have

$$B^*ds \leq c \left(\frac{2R}{R^2 - |w|^2} \right)^p |dw|,$$

where c is a positive constant. Since $\rho < 1$, it then follows that

$$d(0) \leq \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} B^*ds \leq c \int_0^R \left(\frac{2R}{R^2 - |w|^2} \right)^p |dw| < \infty,$$

where $d(0)$ denotes the distance from the origin 0 to the boundary of M . This contradicts the assumption of completeness of M . Hence the proof of the first part of Theorem 1 is complete.

We now prove the second part.

For any complete minimal surface M immersed in R^n , if there are $q > n(n+1)/2$ hyperplanes in general position in CP^{n-1} such that its Gauss map G is ramified over H_j with multiplicity at least e_j for each j and

$$\sum_{j=1}^q (1 - n/e_j) > n(n+1)/2,$$

we are going to prove that M is flat. Since M is flat if and only if its Gauss map is a constant map (see [12]), we only need to prove that G is a constant map.

If G is not a constant map, then we may assume that G is k -nondegenerate and $1 \leq k \leq n-1$. By the first part of the theorem, we have

$$\sum_{j=1}^q (1 - k/e_j) \leq (k+1)(n - k/2 - 1) + n.$$

Since

$$(k+1)(n - k/2 - 1) + n \leq n(n+1)/2,$$

and

$$\sum_{j=1}^q (1 - (n-1)/e_j) \leq \sum_{j=1}^q (1 - k/e_j),$$

we obtain

$$\sum_{j=1}^q (1 - (n-1)/e_j) \leq n(n+1)/2.$$

This contradicts the assumption. Therefore M is flat. Q.E.D.

5. PROOF OF THEOREM 2

Let $x = (x_1, x_2, x_3) : M \rightarrow R^3$ be a nonflat minimal surface and $g : M \rightarrow CP^1$ the Gauss map. Assume $M = \Delta$ (as the argument above). Set $\varphi_i = \partial x_i / \partial z$ ($i = 1, 2, 3$) and $f = \varphi_1 - \sqrt{-1}\varphi_2$. Then according to [12] or [7], the metric on M induced from R^3 is given by

$$(5.1) \quad ds^2 = |f|^2(1 + |g|^2)^2 |dz|^2.$$

Take a reduced representation $\tilde{g} = (g_0, g_1)$ of g on M . Then we can rewrite

$$(5.2) \quad ds^2 = |h|^2 |\tilde{g}|^4 |dz|^2,$$

where $h = f/g_0^2$, and moreover $h \neq 0$. The rest of the steps are the same as the proof of Theorem 1. If M is not flat, then g is not a constant map. Assume that g is ramified over a_j with multiplicity of e_j and $\sum_{j=1}^q (1 - 1/e_j) > 4$, we shall derive a contradiction. Let $P(\alpha_j) = a_j$, $\alpha_j \in C^2$. Consider numbers

$$\rho = \gamma = \frac{1 + 2q/N}{\sum_{j=1}^q (1 - 1/e_j) - 2 - 2q/N},$$

$$\delta = \frac{1}{(1 - \rho) \left(\sum_{j=1}^q (1 - 1/e_j) - 2 - 2q/N \right)}.$$

Choose some N with

$$\frac{\sum_{j=1}^q (1 - 1/e_j) - 3}{3} > 2q/N > \frac{\sum_{j=1}^q (1 - 1/e_j) - 3}{3 + 1/q}$$

so that $0 < 2\rho < 1$, $\frac{2\delta}{N} > 1$. Consider the open subset $M' = M - (\{\tilde{g}_1 = 0\})$ of M and define the function

$$v = h^{1/(1-\gamma)} \left(\frac{\prod_{j=1}^q |(\tilde{g}, \alpha_j)|^{(1-1/e_j-4/N)}}{|\tilde{g}_1|^{1+2q/N}} \right)^\delta$$

on M' where $\tilde{g}_1 = \tilde{g} \wedge \tilde{g}'$.

By exactly the same argument as in the proof of Theorem 1, we can find a curve Γ_{a_0} tends to the boundary of M , and we can estimate the pull-back metric, eventually we obtain that Γ_{a_0} has finite length, contradicting the completeness of the given minimal surface M . Q.E.D.

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